ENPH 344: Final Report

The Classical Limit of the Quantum Harmonic Oscillator

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Introduction

The laws of motion first introduced by Sir Issac Newton in the late 1600s have proven to model systems in our universe with impeccable accuracy. However, in several cases, classical physics has failed to effectively model systems on an extremely small scale. The introduction of Quantum Mechanics works to fill the void where classical physics breaks down, effectively modelling quantum systems that cannot be modelled by Newton's laws of motion. One such system is the quantum harmonic oscillator. The quantum harmonic oscillator can be thought of classically as a simple harmonic oscillator, such as a mass-spring system. Only in the quantum case, one takes the mass in question to be extremely small, such as an oscillating particle or diatomic molecule. When comparing the classical model of the quantum oscillator to the quantum model, it is noticed that the two systems behave in different ways. This is particularly evident when looking at the position probability distributions at low energy states. Figure 1 below shows the probability distribution of an oscillating particle in the ground state as modelled by both quantum and classical mechanics. Notably, the quantum model predicts that the particle is most likely found at the classical equilibrium point, when the position is equal to zero. However, the classical model predicts the exact opposite, that the particle is most likely found far from equilibrium at the point when its position is equal to the amplitude of oscillation. This discrepancy suggests a concerning reality; both classical mechanics and quantum mechanics have proven to be extremely effective at modelling systems in their respective fields, however, the two models do not agree. In order to understand whether this discrepancy is of concern, the two models must be explored as the amplitude of oscillations increases. This brings to light the question; at large amplitudes, does the behaviour of the quantum harmonic oscillator approach the classical behaviour?

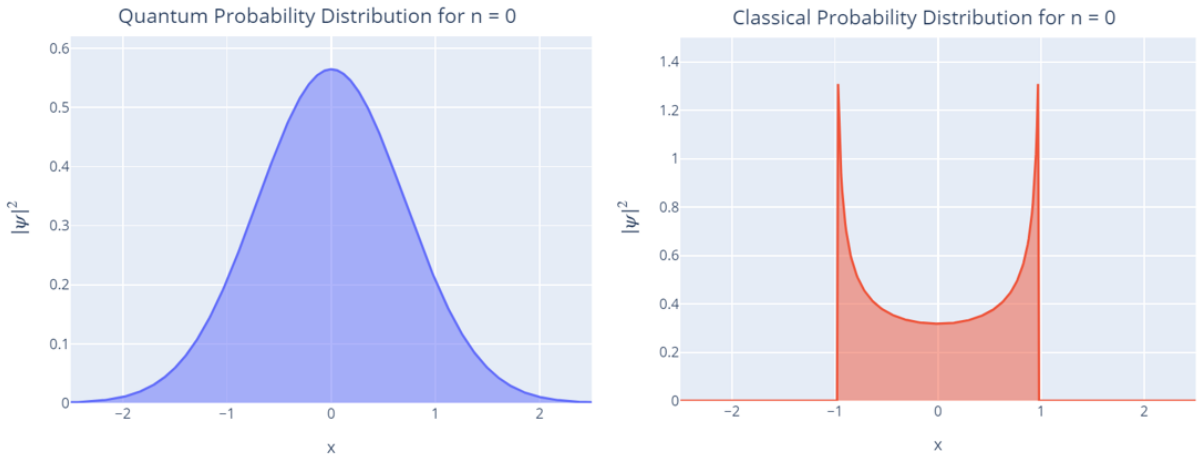

Figure 1: The quantum position probability distribution as compared to the classical probability distribution for the ground state of the quantum harmonic oscillator. Note that the two distributions predict significantly different behaviours of the particle. The code used to generate these plots can be found in Appendix A.

This paper explores the limit of the quantum harmonic oscillator, and inevitably concludes that the position probability distribution of the quantum harmonic oscillator approaches the classical behaviour at large values of the quantum number *n*. In order to determine this correspondence, the probability distributions for both the quantum and classical systems were calculated. In quantum mechanics, the position probability distribution of an oscillating particle can be found by taking the wave function squared. As the solutions to the Schrödinger equation can be found analytically using the set of Hermite polynomials, one can calculate the probability distributions for different values of n. For the classical

model, the probability can be calculated by looking at an infinitely small position “dx”. If the time that the particle spends in that position is an infinitely small time step “dt”, then the probability of finding the particle in the position “dx” is simply dt divided by the total period of oscillation. Integrating over the entire period and normalizing gives an equation for the probability of finding the particle at any arbitrary point “x”. However, its is found that the quantum probability relies on the quantum number “n” while the classical probability relies on the amplitude of oscillation, and as such, there is no necessarily obvious way of comparing these two distributions. This gives rise to the intermittent question; for the quantum oscillator, how is amplitude defined? With further algebraic manipulation, it can be found that, both the classical distribution and quantum distribution rely on the energy E_n of the particle. Regardless of the model, it is understood that the energy of a particle is the same, and as such, both systems consequently rely on the quantum number, “*n*”. Thus, for the purpose of this study, this paper proposes that the quantum amplitude is in fact the quantum number “*n*”. This allows both equations to be written in terms of n. Further, this allows for the position probability distribution plots to be generated, for *both* the classical and quantum oscillator, at increasing values of n. Plots were generated for n values from 0 to 150. For the n = 150 distribution, it was found that the two systems agreed to an error of 1.75%. Comparably to the error at the ground state of 21.21%, it can be concluded that the quantum harmonic oscillator approaches the classical behaviour for large quantum amplitudes.

Model and Method of Approach

In working to generate probability distribution plots, the first objective is to determine the mathematical equations that govern these distributions. This section explores the mathematics in determining the probability distributions for both the quantum and classical distributions, as well as outlines how the proposed quantum amplitude *n* allows us to compare the two systems.

Quantum Probability Distribution

In quantum mechanics, the probability distribution can be found by taking the wave function squared. As such, the wavefunction of the quantum harmonic oscillator must be determined first. This can be done by finding the solutions to the Schrödinger equation. The time independent Schrödinger equation is given by equation 1 below:

$$\frac{-\hbar^2}{2m}\frac{d^2\Psi(x)}{dx^2} + U(x)\Psi(x) = E\Psi(x) \tag{1}$$

The Hamiltonian for the simple harmonic oscillator can be expressed as:

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

Noting that the potential energy of the oscillator is found in the second term of the Hamiltonian, it is found that:

$$U(x) = \frac{1}{2}kx^2$$

Expressing k in terms of the frequency, we have that:

$$\omega = \sqrt{\frac{k}{m}} \rightarrow k = \omega^2 m$$

$$U(x) = \frac{1}{2}\omega^2 m x^2 \tag{2}$$

Subbing equation (2) into equation (1), we have the time dependant Schrödinger equation for the quantum harmonic oscillator:

$$\frac{-\hbar^2}{2m}\frac{d^2\Psi(x)}{dx^2} + \frac{1}{2}m\omega^2 x^2\Psi(x) = E\Psi(x)$$

Rearranging, we have:

$$\frac{-\hbar^2}{2m}\frac{d^2\Psi(x)}{dx^2} + \frac{1}{2}m\omega^2 x^2\Psi(x) - E\Psi(x) = 0$$

Isolating the differential terms:

$$\frac{d^2\Psi}{dx^2} + \frac{\omega^2 x^2\Psi}{-\hbar^2} + \frac{2mE\Psi}{\hbar^2} = 0$$

$$\frac{d^2\Psi}{dx^2} + \left(\frac{2mE}{\hbar^2} - \frac{m^2\omega^2}{\hbar^2}x^2\right)\Psi = 0 \tag{3}$$

Equation (3) is a second order differential equation that can be solved using a power series. Notably, there are other methods to solve this equation, however, as we are looking for the full set of solutions, the power series method seems logical.

To simplify the algebra, first the following change of variables can be made:

$$y = \sqrt{\frac{m\omega}{\hbar}}x$$

Substituting into the differential equation shown in equation (3), we have:

$$\frac{d^2\Psi(y)}{dy^2} + \left(\frac{2E}{\hbar\omega} - y^2\right)\Psi(y) = 0 \tag{4}$$

Note that for extremely large values of y, the y^2 term will dominate over the $\frac{2E}{\hbar\omega}$ term. As such, the ansatz can be made that the general solution to the differential equation will take the form:

$$\Psi(y) = u(y)e^{-\left(\frac{y^2}{2}\right)} \tag{5}$$

Calculating the first derivative by the chain rule:

$$\frac{d\Psi}{dy} = \frac{du}{dy}e^{-\left(\frac{y^2}{2}\right)} - u(y)ye^{-\left(\frac{y^2}{2}\right)}$$

Calculating the second derivative:

$$\frac{d^2\Psi(y)}{dy^2} = \frac{d^2u}{dy^2}e^{-\left(\frac{y^2}{2}\right)} - 2\left(y\frac{du}{dy}e^{-\left(\frac{y^2}{2}\right)}\right) + u(y)(y^2-1)e^{-\left(\frac{y^2}{2}\right)} \tag{6}$$

Subbing equations (5) and (6) into equation (4), we have:

$$\left[\frac{d^2u}{dy^2}e^{-\left(\frac{y^2}{2}\right)} - 2\left(y\frac{du}{dy}e^{-\left(\frac{y^2}{2}\right)}\right) + u(y)(y^2-1)e^{-\left(\frac{y^2}{2}\right)}\right] + \left(\frac{2E}{\hbar\omega} - y^2\right)u(y)e^{-\left(\frac{y^2}{2}\right)} = 0$$

Cancelling terms and dividing through by $e^{-\left(\frac{y^2}{2}\right)}$, we have:

$$\frac{d^2u}{dy^2} - 2y\frac{du}{dy} + \left(\frac{2E}{\hbar\omega} - 1\right)u(y) = 0 \tag{7}$$

Equation (7) now represents the time dependant Schrödinger equation in terms of the changed variable y. To solve this second order differential equation, we can use the power series method, making the ansatz that:

$$u(y) = \sum_0^\infty a_n y^n \tag{8}$$

Taking the first derivative:

$$\frac{du}{dy} = \sum_0^\infty n a_n y^{n-1} \tag{9}$$

And the second derivative:

$$\frac{d^2u}{dy^2} = \sum_0^\infty n(n-1)a_n y^{n-2} = \sum_0^\infty (n+1)(n+2)a_{n+2}y^n \tag{10}$$

Subbing equations (8), (9) and (10) into equation (7):

$$\sum_{0}^{\infty}(n+1)(n+2)a_{n+2}y^{n} - 2y\sum_{0}^{\infty}na_{n}y^{n-1} + \left(\frac{2E}{\hbar\omega} - 1\right)\sum_{0}^{\infty}a_{n}y^{n} = 0$$

Simplifying:

$$\sum_{0}^{\infty}\left[(n+1)(n+2)a_{n+2} - \left(\frac{2E}{\hbar\omega} - 1 - 2n\right)a_{n}\right]y^{2n} = 0$$

Thus, the recursion relation can be defined as:

$$a_{n+2} = \frac{2n+1-\frac{2E}{\hbar\omega}}{(n+2)(n+1)}a_{n} \tag{11}$$

Equation (11) defines that series representation of all the expansion coefficients for the solution to equation (7). In order to represent the series in closed form, we can set the numerator of equation (11) to zero in order to solve for the energy in terms of n:

$$2n + 1 - \frac{2E}{\hbar\omega} = 0$$

$$E = \left(n + \frac{1}{2}\right)\hbar\omega \tag{12}$$

Equation (12) represents the set of energies for each quantum number *n*. Although the power series solution is not in closed form, with further manipulation beyond the scope of this course, a closed solution can be found by incorporating Hermite Polynomials. The Hermite Polynomials are a set of orthogonal polynomials that are closely related to the recursion relation defined in equation (11) [1]. Applying these polynomials, the general solutions to equation (13) can be written as:

$$\Psi(y) = \left(\frac{y^{2}}{x^{2}\pi}\right)^{\frac{1}{4}}\frac{1}{\sqrt{2^{n}n!}}H_{n}(y)e^{-\left(\frac{y^{2}}{2}\right)}$$

Taking into consideration our change of variables, we have the general formula for the normalized wave function of the quantum harmonic oscillator to be:

$$\Psi(x) = \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}}\frac{1}{\sqrt{2^{n}n!}}H_{n}\left(\sqrt{\frac{m\omega}{\hbar}}x\right)e^{-\left(\frac{m\omega}{2\hbar}x^{2}\right)} \tag{13}$$

Further, the probability distribution can be found by taking the wavefunction squared:

$$P_{q}(x) = |\Psi(x)|^{2} \quad s.t. \quad \Psi(x) = \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}}\frac{1}{\sqrt{2^{n}n!}}H_{n}\left(\sqrt{\frac{m\omega}{\hbar}}x\right)e^{-\left(\frac{m\omega}{2\hbar}x^{2}\right)} \tag{14}$$

Equation (14) represents the position probability distribution for the quantum harmonic oscillator, dependant on the quantum number n.

Classical Probability Distribution

Deriving the probability distribution for the classical harmonic oscillator requires a more creative strategy. First, we propose analyzing the probability of finding the particle in an infinitesimally small range between an arbitrary value “x” and a step “dx”.

$P(x)dx = probability\ of\ finding\ a\ particle\ between\ x\ and\ x + dx$

For a simple harmonic oscillator, the period is given by:

$$T = \frac{2\pi}{\omega}$$

Consequently, it follows that the probability of finding a particle between x and x + dx would simply be the amount of time that the particle spends in the range, divided by the total period of oscillation. Letting "dt" be the amount of time the particle spends in the range x + dx, we have:

$$P(x)dx = \frac{dt}{T} \tag{15}$$

Note that the time step, "dt", is related to the velocity "v" such that:

$$v = \frac{dx}{dt}$$

It follows that:

$$dt = \frac{dx}{v} \tag{16}$$

Subbing equation (16) into equation (15), and applying the formula for period we have:

$$P(x)dx = \frac{dx}{vT} = \frac{\omega}{2\pi v}dx \tag{17}$$

Under classical mechanics, energy must be conserved. The total energy of a classical harmonic oscillator is given by:

$$E = \frac{1}{2}mv(t)^2 + \frac{1}{2}m\omega^2 x(t)^2 \tag{18}$$

Rearranging for velocity, we have:

$$v(t) = \sqrt{\frac{2E}{m} - \omega^2 x(t)^2} \tag{19}$$

Recall that the position in time of a harmonic oscillator as described by classical mechanics is:

$$x(t) = Asin(\omega t + \phi), \qquad s.t.\ \ \omega = natural\ frequency,\ A = amplitude,\ \phi = phase$$

Taking ϕ to be equal to -$\frac{\pi}{2}$ to simplify the manipulation, we have that:

$$x(t) = Asin(\omega t) \tag{20}$$

Subbing equation (20) into equation (19), we have that:

$$v(t) = \sqrt{\frac{2E}{m} - \omega^2 A^2 \sin^2(\omega t)} \tag{21}$$

Note that the Amplitude, A, of a harmonic oscillator is given by:

$$A = \sqrt{\frac{2E}{m\omega^2}}$$

Thus,

$$A^2 = \frac{2E}{m\omega^2} \rightarrow \omega^2 A^2 = \frac{2E}{m} \quad (22)$$

Subbing equation (22) into equation (21) gives:

$$v(t) = \sqrt{\omega^2 A^2 - \omega^2 A^2 \sin^2(\omega t)} = \omega\sqrt{A^2 - A^2 \sin^2(\omega t)}$$

Further noting that the second term under the square root is equivalent to position equation squared:

$$v(t) = \omega\sqrt{A^2 - x(t)^2} \quad (23)$$

Thus, returning to our equation for probability, equation (17), we have that:

$$P(x)dx = \frac{\omega}{2\pi v}dx = \frac{1}{2\pi\sqrt{A^2 - x(t)^2}}dx$$

Normalizing we have the condition that:

$$\int_{-\infty}^{\infty} P(x)dx = 1$$

Which gives the equation for the probability distribution of the classical harmonic oscillator:

$$P_C(x) = \frac{1}{\pi\sqrt{A^2 - x^2}}, \quad s.t. \quad A^2 = \frac{2E}{m\omega^2} \quad (24)$$

Determining Quantum Amplitude

Effectively, we now have two equations representing the probability of finding the particle at a particular location "x" as described by both quantum mechanics, equation (14), and classical mechanics, equation (24). However, note that equation (14) depends on the quantum number "n" whereas equation (24) depends on our classical amplitude "A". Referring back to our guiding question, we hope to explore the behaviour of the quantum harmonic oscillator at "large amplitudes". As such, we need to define what is meant by "amplitude" at the quantum level so that we can reconcile the two distributions.

As with many methods in this course, a proposition can be made using the classical case to see if it logically holds for the quantum case. In classical mechanics, the amplitude can be described by equation (22), which is simply a constant multiplied by the energy of the particle:

$$A^2 = \frac{2E}{m\omega^2} = (const.)E$$

As an extension, we make the proposition that the "quantum amplitude" may follow suit:

$$A_n^2 = (const.)E_n$$

Notably, both the classical amplitude and our proposed quantum amplitude are solely dependant on energy. Regardless of whether the classical or quantum model is being considered, we know the energy of a particle to be:

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) \quad (26)$$

Further, we note that the energy, E, is dependant on a series of constants and *n*, the quantum number. Thus, we can narrow down the original proposition and suggest that the quantum number "n" is representative of our quantum "amplitude". This allows us to express equation (24), the probability for the classical distribution, in terms of n.

Subbing equation (22) into equation (24), we have:

$$P_C(x) = \frac{1}{\pi\sqrt{\frac{2E}{m\omega^2} - x^2}}$$

Subbing in equation (26):

$$P_C(x) = \frac{1}{\pi\sqrt{\frac{2\hbar\left(n+\frac{1}{2}\right)}{m\omega} - x^2}} \tag{27}$$

Confirming the proposition for quantum amplitude

The proposition that *n* is representative of the "quantum amplitude" can be confirmed by comparing the behaviour of the known classical amplitude to the proposed quantum amplitude. Classically, amplitude can be defined as the maximum displacement of an oscillation measured from the equilibrium point. Consequently, for very small amplitudes, we would have a high probability of finding the particle close to the equilibrium point. On the contrary, for large amplitudes, we see that there is a higher probability of finding the particle further from the equilibrium point. Relating this to a quantum system, for small n, we know that there is a higher probability of finding the particle close to the equilibrium point. This is shown by the position distribution for n = 0 in Figure 2 below. Similar to the classical system, as our proposed amplitude "n" increases, we see an increasing probability that the particle will be found further from the equilibrium point. Thus, graphically, both the classical amplitude and proposed quantum amplitude behave similarly, suggesting that n is a sound proposition for the quantum amplitude.

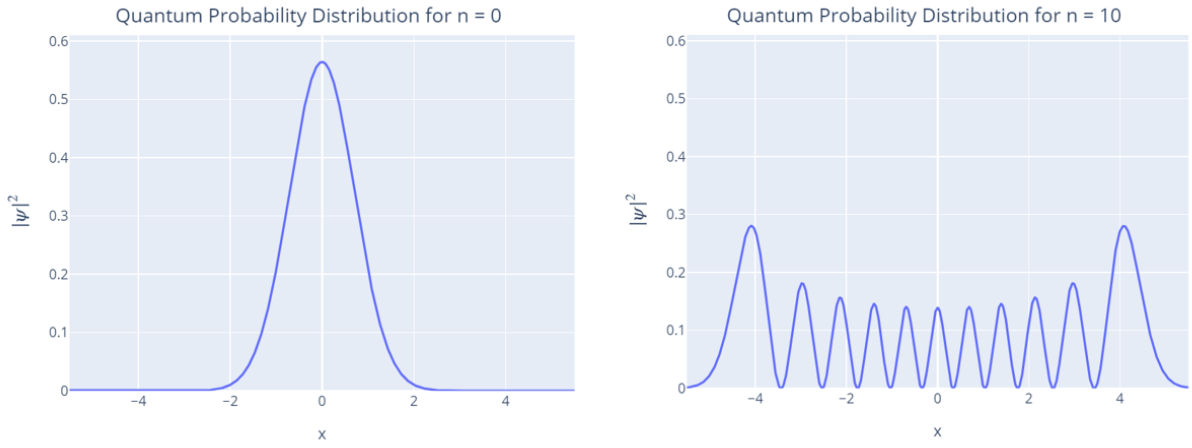

Figure 2: Quantum probability distribution for the quantum amplitude of n = 0 and n = 10. Note that, similarly to the classical amplitude, as n increases, there is a greater probability of finding the particle further from the equilibrium point.

Results and Discussion

Using equations (14) and (27), plots for both distributions were generated for values from n = 0 to n = 150. By superimposing the classical distribution on top of the quantum distributions, we can explore the qualitative behaviour of the distributions to inspect if the quantum distribution approaches the classical distribution at large amplitudes. Further, we can quantitatively define a standard error between the two distributions to better understand the significance of the results.

Generating Plots

First, a series of plots were generated for low values of n. For simplicity, all constant variables were taken to be 1. Figure 3 below shows the classical distribution, as modelled by equation (27) superimposed on the quantum distribution, modelled by equation (14), across varying values of n.

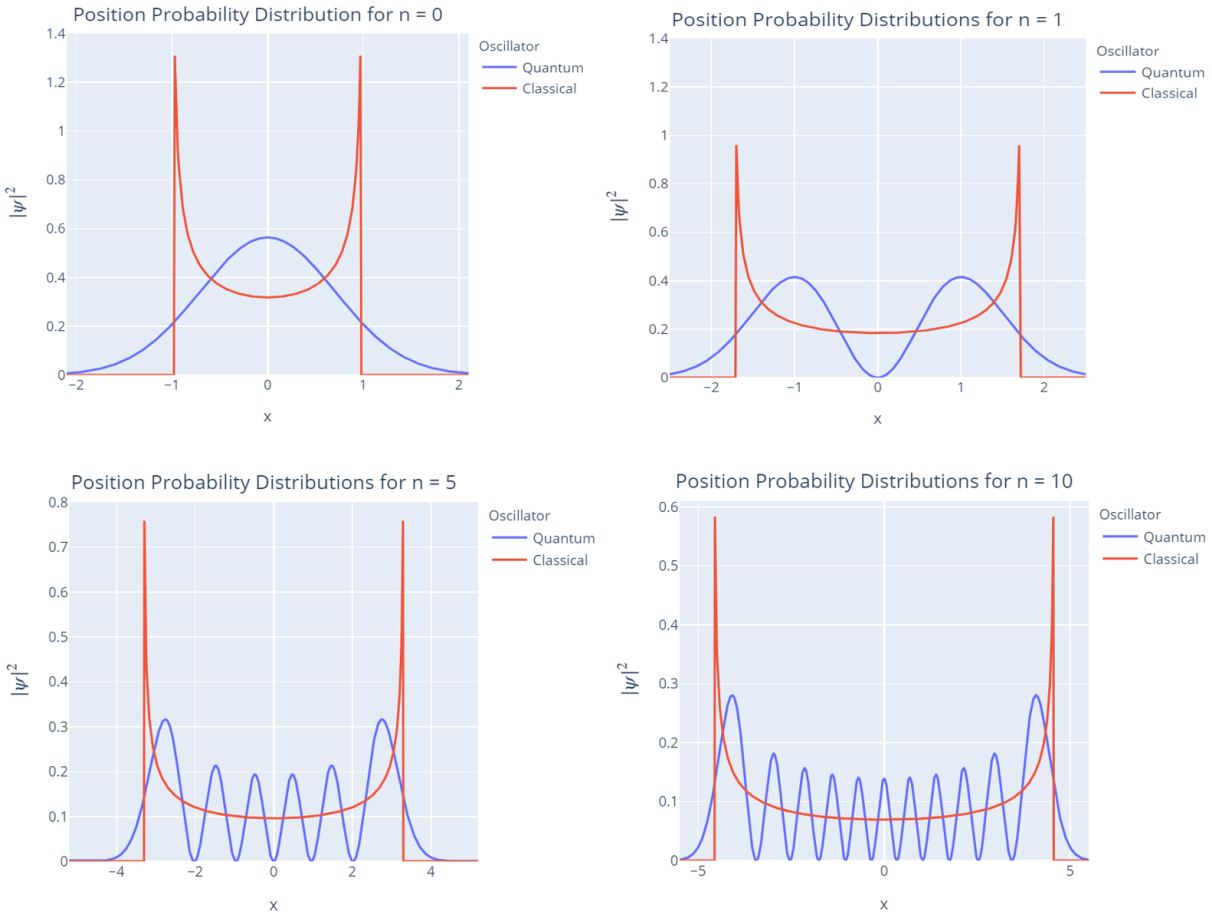

Figure 3: Plots comparing the position probability distributions for the classical and quantum models of the harmonic oscillator. Note that at the lower values of n, ie) n =0,1, there is no obvious correspondence between the classical and quantum models, however, as n increases, a slight correspondence is observed. These plots were generated with python code, using the plotly express plotting library, attached in Appendix A.

Note that, for lower values of n, there is no obvious correlation between the two distributions. In fact, for the n = 0 case, the distributions are predicting the exact opposite behaviour in that the quantum case suggests that the particle is *most* likely to be found at the point x = 0, while the classical case suggests that the particle is *least* likely to be found at the point x = 0. However, as n begins to increase, it

becomes apparent that the quantum distribution begins to exhibit similarities to the classical distribution. Particularly, for the n = 10 case, it can be seen that the particle is now more likely to be found at points further from the equilibrium.

To confirm this holds, plots were explored at larger values of n, as shown in Figure 3 below.

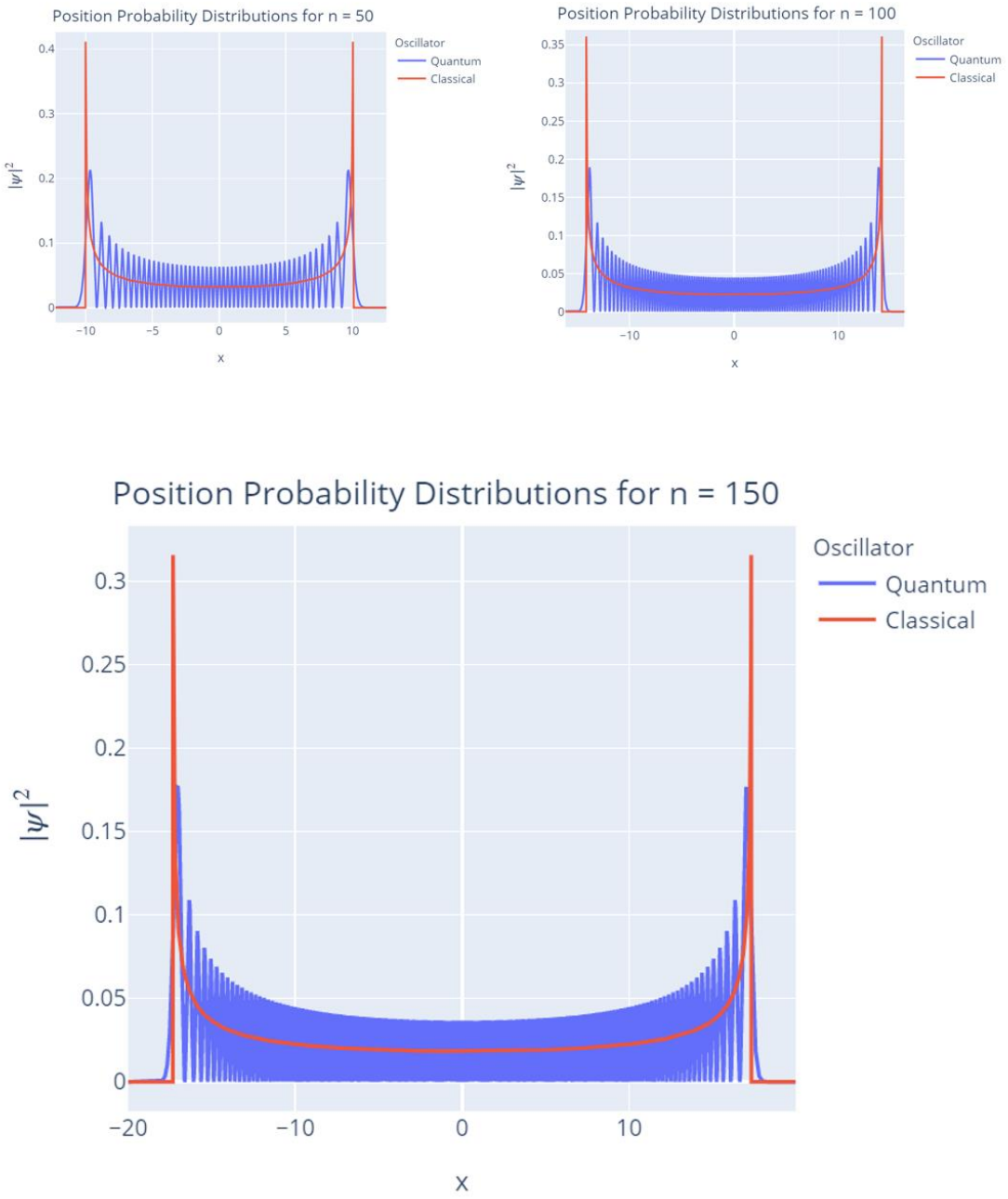

Figure 4: Plots comparing the position probability distributions for the classical and quantum models of the harmonic oscillator. Note that as the value of n increases, we begin to see an evident correspondence between the distributions. In particular, we see that as n increases, the frequency of oscillations also increases. As well, notice that the probability of finding the particle outside the classically allowed region, that is beyond the point x = A, decreases as n increases.

Notably, as n increases, we see a striking correspondence between the quantum distribution and the classical distribution. In particular, there are two key properties that can be witnessed. First, we see that as n increases, the frequency of oscillations in the quantum distribution also increases. At points where the quantum distribution intersects the classical distribution, we have that both distributions predict the exact same probability. Thus, more frequent oscillations suggest that there are more intersections between the two curves, and therefore more points in x at which the two distributions agree. Secondly, we see that as n increases, the probability of finding the particle outside the classically allowed region, that is beyond the point x = A, decreases.

Extrapolating these findings to larger values of n, we assume that, in the limit that n approaches infinity, the frequency of oscillations becomes so great that the individual oscillations cannot be resolved from one another, thus, for all values of x, we would have both distributions predicting the same probability. As well, we assume that in this limit, the probability of finding the particle outside of the classically allowed region approaches zero. Therefore, to answer the guiding question of this paper with qualitative evidence, at large amplitudes, the behaviour of the position probability distribution as modelled by quantum mechanics approaches the behaviour of the classical model

Error Calculations

Notably, the distributions show a number of qualitative similarities. However, we can also quantify these similarities by looking at the percent error when comparing to how the quantum distribution models the classical distribution.

For any arbitrary value “x”, the percent error can be calculated by:

$$\%\ error(x) = \left|\frac{P_q(x) - P_C(x)}{P_{c(x)}}\right| * 100\% \tag{28}$$

From equation (28), the error for each distinct value of x can be calculated. Summing all errors and dividing by the total number of records gives the average error for any particular value of n:

$$Avg\ \%\ Error = \frac{\sum_{x_0}^{x_0+L}\left|\frac{P_q(x) - P_C(x)}{P_{c(x)}}\right| * 100\%}{\#\ of\ records} \tag{29}$$

Figure 5 below shows the average error calculated for each value of n from 0 to 150.

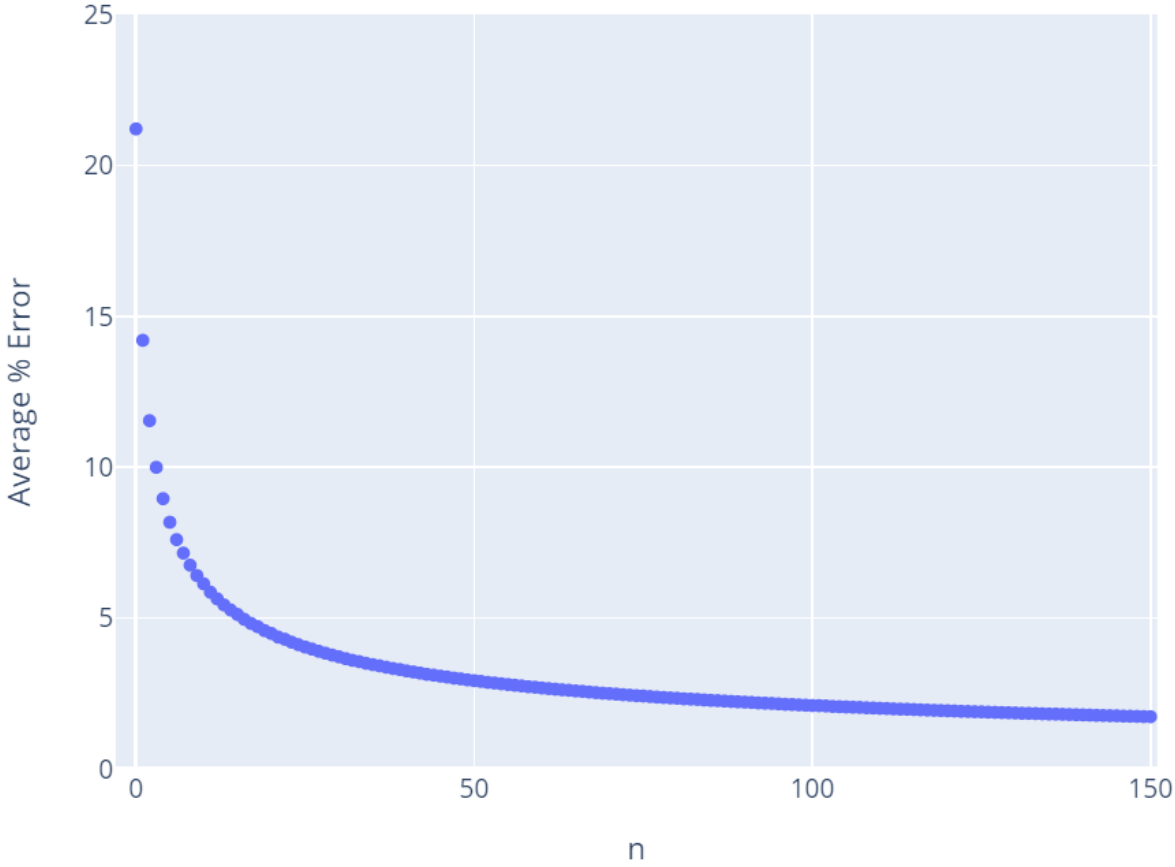

Figure 5: The average % error when comparing the position probability as predicted by quantum mechanics to classical mechanics. Note that as n increases, we see an exponential decrease in the average percent error. This suggests that as n increases, the error between the quantum probability and classical probability decreases, and thus, the quantum model more closely approaches the classical model. The code used to generate this plot can be found in Appendix A

Notably, as n increases, we see an exponential decrease in the average percent error between the two distributions. For the n = 0 case, the average percent error was determined to be 21.21%. However, for the n = 150 case, the percent error drops to 1.74%. In plain English, this suggests that, for any arbitrary value of x, the quantum model will predict the probability within 21.21% of the classical model, on average. Whereas for the n = 150 case, this error reduces to 1.74%, almost 13 times more accuract. As this exponential trend continues, we assume that in the limit as n approaches infinity, the average percent error approaches zero. In the case where the percent error is equal to zero, we have that the quantum model and classical model would predict the exact same outcomes. Therefore, we can conclude that quantitatively, the quantum model approaches the classical model in the limit of large amplitudes.

Justification of results

Due to both the qualitative evidence shown by the visual similarities between the graphs, and the quantitative evidence shown by the reducing average error, it was concluded that the behaviour of the quantum model approaches the classical model in the limit of large amplitudes. However, as with any result in a scientific study, it is important to reflect on the logic of the outcome. Notably, Newton's laws

have proven to be extremely effective at modelling classical systems, and likewise, quantum mechanics is extremely effective at modelling the behaviour of quantum particles. As a result, we anticipate that, since both are proven at modelling the oscillator at their respective scales, there must be a correspondence between the two. Neil's Bohr introduced the concept of the "correspondence principle" when analyzing the radiation of excited atoms as described by quantum mechanics compared to the radiation modelled by classical mechanics. The principle states that the behaviour of systems described by the theory of quantum mechanics reproduces classical physics in the limit of large quantum numbers [2]. As such, it is reassuring that we see the quantum behaviour approach the classical behaviour for large values of n. If this was not the case, it would likely suggest an inconsistency in either or both models.

Conclusion

With the support of both qualitative and quantitative results, it can be concluded that the behaviour of the quantum harmonic oscillator approaches the behaviour of the classical harmonic oscillator in the limit of large amplitudes. In particular, for values of the quantum number *n* from 0 to 150, plots were generated in order to explore the superimposition of the classical position distribution on the quantum position distribution. Three key outcomes were observed as n increased; firstly, it was noted that the frequency of oscillations in the quantum distribution increased, suggesting that the two distributions predicted the same results more frequently, secondly, the probability of finding a particle outside the classically allowed region decreased, and thirdly, the average percent error between distributions decreased exponentially. Taken together, these results suggest that when extrapolated to larger values of n, the quantum distribution would reproduce the model described by classical physics. Although this result is confined to the problem of the harmonic oscillator, it suggests the deeper implication that quantum physics can be reconciled with classical physics. This leads to the indication that quantum mechanics may be able to effectively model macroscopic systems, not only limited to the harmonic oscillator. The idea that macroscopic systems can potentially be described by a wavefunction and a set of probability distributions challenges classical physics and its deterministic nature.

Although unsettling, this brings to light several areas for further investigation. With regards to the quantum harmonic oscillator, it would be particularly beneficial to expand this study to the three-dimensional case and explore whether probability orbitals or atoms replicate classical physics in the limit of large quantum numbers. Additionally, the exploration of a rigorous mathematical proof exploring the behaviour of the probability functions in the limit that n approaches infinity would solidify the findings of this paper. Beyond the harmonic oscillator, investigating Bohr's correspondence principle with regards to other problems in quantum mechanics would help to confirm these results. In doing so, one would effectively suggest that, quantum mechanics can take the place of classical mechanics in modelling macroscopic systems and further solidify quantum mechanics as a fundamental theory in physics.

References

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[2] Wikipedia, "Correspondence Principle," Wikipedia, 2020. [Online]. Available: https://en.wikipedia.org/wiki/Correspondence_principle.

[3] R. Nave, Time Dependent Schrodinger Equation, http://hyperphysics.phy-astr.gsu.edu/hbase/quantum/Scheq.html.

Appendix A: Python Code for Figure Generation

The following code was used to generate the data frames from which the distribution graphs were plotted from

In [42]:

```
import math as math
import numpy.polynomial.hermite as Herm
import pandas as pd

m=1
w=1.0
hbar=1.0
#CHOOSE PLOT RANGE
dx = 0.01
x_lim = 20
nMax = 50
n_list = [0,10]

def hermite(x, n):
     xi = numpy.sqrt(m*w/hbar)*x
     herm_coeffs = numpy.zeros(n+1)
     herm_coeffs[n] = 1
     return Herm.hermval(xi, herm_coeffs)

def quantum(x,n):
    xi = numpy.sqrt(m*w/hbar)*x
    prefactor = 1./math.sqrt(2.**n * math.factorial(n)) * (m*w/(numpy.pi*hbar))**(0.25)
    psi = prefactor * numpy.exp(- xi**2 / 2) * hermite(x,n)
    return psi

def classical(x,n):
    E = hbar*w*(n+0.5)
    x_max = numpy.sqrt(2*E/(m*w**2))
    classical_prob = numpy.zeros(x.shape[0])
    x_inside = abs(x) < (x_max - 0.025)
    classical_prob[x_inside] = 1./numpy.pi/numpy.sqrt(x_max**2-x[x_inside]*x[x_inside])
    return classical_prob

data = []
cols = []
df2 = pd.DataFrame(data = data, columns = cols)
del df2
df2 = pd.DataFrame(data = data, columns = cols)
x_list = numpy.arange(-x_lim,x_lim,dx)
classic = ['Classical']*len(x_list)
quantum = ['Quantum']*len(x_list)
for n in n_list:
    n_temp = []
    psi_list = []
    for x in x_list:
        df_temp = pd.DataFrame(data= data, columns = cols)
        psi_list.append(quantum(x,n)*quantum(x,n))
        n_temp.append(n)
    df_temp['Oscillator'] = quantum
    df_temp['n']=n_temp
    df_temp['x']=x_list
    df_temp['$|\psi|^2$']=psi_list
    df2 = df2.append(df_temp)
    del df_temp
    ### Now appending the classical case
    df_temp = pd.DataFrame(data= data, columns = cols)
    df_temp['Oscillator'] = classic
    df_temp['n']=n_temp
    df_temp['x']=x_list
    df_temp['$|\psi|^2$']=classical(x_list,n)
    df2 = df2.append(df_temp)
    del df_temp
df2
```

The following code was used to generate the distribution plots

```
import plotly.express as px

fig = px.area(df2, x="x", y='$|\psi|^2$', animation_frame="n",
           color="Oscillator", hover_name="Oscillator",
           #title="n = 0",
           range_x=[-2.5,2.5], range_y=[0,0.62])
fig.update_layout(
    title={
        'text': "Quantum Probability Distribution for n = 0",
        'y':0.93,
        'x':0.45,
        'xanchor': 'center',
        'yanchor': 'top'})
fig.show()
```

The following code was used to generate the error values for each n value.

```
m=1
w=1.0
hbar=1.0
#CHOOSE PLOT RANGE
dx = 0.01
nMax = 151
n_list = np.arange(0,nMax,1)
errList = []

data = []
cols = []
df2 = pd.DataFrame(data = data, columns = cols)
del df2
df2 = pd.DataFrame(data = data, columns = cols)
for n in n_list:
    x_lim = numpy.sqrt(2*hbar*w*(n+0.5)/(m*w**2))
    x_list = numpy.arange(-x_lim,x_lim,dx)
    classic = ['Classical']*len(x_list)
    quantum = ['Quantum']*len(x_list)
    n_temp = []
    psi_list = []
    for x in x_list:
        df_temp = pd.DataFrame(data= data, columns = cols)
        psi_list.append(quantum(x,n)*quantum(x,n))
        n_temp.append(n)
    df_temp['Oscillator'] = quantum
    df_temp['n']=n_temp
    df_temp['x']=x_list
    df_temp['$|\psi|^2$']=psi_list
    df_temp["Classical Prob"] = classical(x_list,n)
    df_temp["absError"] = np.absolute(df_temp["Classical Prob"]-df_temp['$|\psi|^2$'])
    df_temp["% Error"] = np.absolute(((df_temp["Classical Prob"]-df_temp['$|\psi|^2$'])/df_temp['$|\psi|^2$'])*100)
    AverageError = df_temp['absError'].describe()['mean']*100
    print("For n = " +str(n) + ", Average Error = " + str(AverageError))
    del df_temp
    errList.append(AverageError)
errList
```